A Stabilization Framework for the Output Regulation of Rational Nonlinear Systems

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***Abstract*—A systematic stabilization approach is provided for systems whose regulation error dynamics is subject to rational nonlinearities given prior knowledge of the system zero-error steady-state condition and a proper internal model. The error dynamics is cast in a differential-algebraic form so as to address the synthesis of controller parameters by a numerical optimization problem subject to bilinear matrix inequality constraints. A particular case is also explored where the resulting constraints are linear matrix inequalities.**

***Index Terms*—Differential-algebraic representation, linear matrix inequalities, output regulation, rational nonlinear systems.**

I. Introduction

Ensuring robust output tracking of reference signals and rejection of exogenous perturbations is a fundamental control engineering problem with several practical applications, for instance, power converters [1], spacecraft attitude control [2], trajectory tracking of underwater vehicle-manipulator systems [3], and wind turbines [4]. A remarkable theoretical contribution in this field was the formulation of the regulator equations [5], a set of partial-differential equations characterizing the steady-state of nonlinear systems. General guidelines have also been proposed in order to approach the nonlinear regulator design, which are based on two distinct steps [6]. In the first one, a zero-error and invariant center manifold should be determined using the regulator equations and a proper internal model should be designed in order to generate the steady-state control signal. Afterward, a stabilization problem should be cast so as to ensure attractiveness of the system trajectories with respect to the center manifold. There is, however, no solution entirely systematic and general for both of these steps, as a result, the state of the art on output regulation theory is composed by a myriad of methods [7]–[10]. These studies are focused on different classes of problems, on different ways to design internal models capable of handling parametric uncertainties and also on different procedures to robustly address the system stabilization.

A well-established theory for control design and stability analysis of nonlinear systems, albeit not explored in the output regulation context, is the differential-algebraic representation [11], which is able to systematically account for rational nonlinearities (products and quotients of polynomial functions). Important contributions in this field include—a method for estimating the stability region of nonlinear systems with input saturation [12], antiwindup design for a class of nonlinear systems [13], and local input-to-state stabilization of quadratic systems [14].

This article presents a new systematic stabilization framework for the output regulation of rational nonlinear systems. We are combining here the differential-algebraic methodology with the classical output regulation theory for nonlinear systems, both important topics which have, so far, not been studied together. In comparison to previous works that used the differential-algebraic method, we deal with new aspects arising from the output regulation problem, such as the internal model stage, the nonvanishing property of exogenous perturbations, and output feedback restrictions for maintaining the desired system steady state. On the other hand, in comparison to previous nonlinear output regulation studies, our framework deals with rational nonlinearities in the regulation error dynamics and does not require commonly assumed plant structures. As a result, our approach is able to systematically design stabilizing controllers to a class of systems which are not necessarily in the normal form. A preliminary development to our work was presented in [15], where a stability analysis method was cast for output regulation systems with rational nonlinearities. This article extends [15] and contributes to the synthesis framework in order to achieve stabilization of the closed-loop system.

Notation: $\mathcal{H}\{A\}$ denotes a symmetric block $A + A^T$. Symmetric elements in a matrix are represented by $(\star)$, whereas $(\cdot)$ hides irrelevant elements. $A \succ 0$ means that a symmetric matrix A is positive-definite. $\mathrm{diag}\{A, B\}$ denotes a diagonal matrix obtained by the elements A and B. $\mathrm{Co}\{\mathcal{V}\}$ is a convex hull formed by the set of vertices $\mathcal{V}$.

II. Preliminaries

Consider a nonlinear system represented by[1]

$$\begin{cases} \dot{x} = f(x, w, u) \\ y = g(x, w) \\ e = h(x, w) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the system state vector, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the output measurement, and $e \in \mathbb{R}^{n_e}$ is the output error. The exogenous disturbance signal $w \in \mathbb{R}^{n_w}$ is assumed to be generated

[1]Parametric uncertainties in (1) can always be regarded as additional exogenous states satisfying $\dot{w} = 0$.

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by the following nonlinear exosystem:

$$\dot{w} = s(w). \tag{2}$$

The system control input is supplied by a nonlinear output feedback controller of the form

$$\begin{cases} \dot{\xi} = \phi(\xi, y) \\ u = \theta(\xi, y) \end{cases} \tag{3}$$

where $\xi \in \mathbb{R}^{n_\xi}$ is the controller state vector. The following preliminary definitions and assumptions are considered.

Definition 1: The closed-loop system (1)–(3) is said to achieve *output regulation* in region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w}$ if $\exists\, \epsilon_1, \epsilon_2, \epsilon_3 > 0 :$ $\|x(t)\| \leq \epsilon_1\,, \|\xi(t)\| \leq \epsilon_2\,, \|w(t)\| \leq \epsilon_3 \,\forall\, t \geq 0$, and $\lim_{t\to\infty} e(t) = 0 \;\forall\,(x(0), \xi(0), w(0)) \in \mathcal{D}$.

Assumption 1: Nonlinear functions in (1)–(3) satisfy $f(0,0,0) = 0, g(0,0) = 0, h(0,0) = 0, s(0) = 0, \phi(0,0) = 0$, and $\theta(0,0) = 0$.

Assumption 2: There exist a known set $\mathcal{W} \subset \mathbb{R}^{n_w}$ such that $w(t) \in \mathcal{W}\,\forall\, t > 0$ if $w(0) \in \mathcal{W}$.

The main control design problem dealt in this article is described as follows.

Problem 1: Design controller functions $\phi(\xi, y)$ and $\theta(\xi, y)$ such that the closed-loop system (1)–(3) achieves output regulation in some region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathcal{W}$.

In order to solve Problem 1, we consider $\xi \triangleq [\xi_m^T \;\; \xi_s^T]^T$, where $\xi_m \in \mathbb{R}^{n_m}$ represents internal model states and $\xi_s \in \mathbb{R}^{n_s}$ denotes stabilizing states. The dynamics of the internal model stage is introduced as

$$\begin{cases} \dot{\xi}_m = \phi_m(\xi_m, y) + v_m \\ u = \theta_m(\xi_m, y) + v_u \end{cases} \tag{4}$$

where $v_u \in \mathbb{R}^{n_u}$ is the system stabilizing input and $v_m \in \mathbb{R}^{n_m}$ is the internal model stabilizing input. In turn, these signals $v \triangleq [v_u^T \;\; v_m^T]^T \in \mathbb{R}^{n_v} (n_v = n_u + n_m)$ are supplied by a stabilizing controller of the form

$$\begin{cases} \dot{\xi}_s = \phi_s(\xi_s, \xi_m, y) \\ v = \theta_s(\xi_s, \xi_m, y). \end{cases} \tag{5}$$

Based on the fundamental nonlinear output regulation result from [5], Lemma 1 defines sufficient conditions for output regulation of the closed-loop system (1), (2), (4), (5).

Lemma 1: [5] The closed-loop system (1), (2) with controller (4), (5) achieves output regulation in $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_s} \times \mathcal{W}$ if there exist smooth mappings $\pi : \mathcal{W} \to \mathbb{R}^{n_x}, c : \mathcal{W} \to \mathbb{R}^{n_u}, d : \mathcal{W} \to \mathbb{R}^{n_y}$ and $\sigma : \mathcal{W} \to \mathbb{R}^{n_m}, \pi(0) = 0, c(0) = 0, d(0) = 0, \sigma(0) = 0$, such that $\forall\, w \in \mathcal{W}$

$$\begin{cases} \dfrac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w), w, c(w)) \\ d(w) = g(\pi(w), w) \\ 0 = h(\pi(w), w) \end{cases} \tag{6}$$

$$\begin{cases} \dfrac{\partial \sigma(w)}{\partial w} s(w) = \phi_m(\sigma(w), d(w)) \\ c(w) = \theta_m(\sigma(w), d(w)) \end{cases} \tag{7}$$

$$\begin{cases} 0 = \phi_s(0, \sigma(w), d(w)) \\ 0 = \theta_s(0, \sigma(w), d(w)) \end{cases} \tag{8}$$

and also $\forall\,(x(0), \xi_m(0), \xi_s(0), w(0)) \in \mathcal{D}$

$$\begin{cases} \lim_{t\to\infty} \|x(t) - \pi(w(t))\| = 0 \\ \lim_{t\to\infty} \|\xi_m(t) - \sigma(w(t))\| = 0 \\ \lim_{t\to\infty} \|\xi_s(t)\| = 0. \end{cases} \tag{9}$$

We assume there exist known solutions $\pi(w), c(w)$, and $d(w)$ with respect to condition (6). Moreover, we consider that feasible internal model functions $\phi_m(\xi_m, y)$ and $\theta_m(\xi_m, y)$ are known such that (7) is satisfied for some $\sigma(w)$. Guidelines on how to perform these traditional preliminary steps are thoroughly explained in [16].

To completely solve Problem 1, it is furthermore necessary to design proper stabilizing functions $\phi_s(\xi_s, \xi_m, y)$ and $\theta_s(\xi_s, \xi_m, y)$ satisfying (8) and the attraction conditions in (9). This last step is the main focus of this article, for which we propose a novel systematic stabilization framework to be exposed in the following section.

III. Proposed Stabilization Framework

A systematic methodology capable of synthesizing stabilizing stages for output regulation, such as (5), is presented in this section. The proposed control architecture will be first presented in Section III-A followed by the development of regulation error dynamics in Section III-B. At last, Section III-C will show stability conditions to ensure manifold attraction.

A. Stabilizing Controller Structure

Let us consider a proxy error signal $\varepsilon \in \mathbb{R}^{n_\varepsilon}, \varepsilon = \delta(y), \delta : \mathbb{R}^{n_y} \to \mathbb{R}^{n_\varepsilon}$, which encompasses the stabilizing feedback information from the measurements to be applied in the controller. In order to address condition (8), $\delta(y)$ can be any function satisfying the following relation:

$$0 = \delta(d(w)) \,\forall\, w \in \mathcal{W} \tag{10}$$

which simply denotes that $\delta(y)$ must vanish to zero when composed with $d(w)$ from (6) $\forall\, w \in \mathcal{W}$.

Remark 1: The role of proxy error ε is to deal with cases where the original output error e is not directly implementable with the output measurements y. In the particular scenario where e is implementable with y, i.e., $\exists\, \mathrm{h}(y) : \mathrm{h}(g(x, w)) = h(x, w)$, then $\delta(y) = \mathrm{h}(y)$ can be considered, in which case $\varepsilon = e$.

Toward devising a numerical procedure able to design the functions $\phi_s(\xi_s, \xi_m, y)$ and $\theta_s(\xi_s, \xi_m, y)$ in (5), it is natural to consider a particular structure with free decision parameters. In this sense we propose the form

$$\begin{cases} \dot{\xi}_s = \mathrm{F}(\xi_m, y)\, \xi_s + \mathrm{G}(\xi_m, y)\, \varepsilon \\ v = \mathrm{H}(\xi_m, y)\, \xi_s + \mathrm{K}(\xi_m, y)\, \varepsilon \end{cases} \tag{11}$$

with matrix functions $\mathrm{F} : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_s \times n_s}, \mathrm{G} : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_s \times n_\varepsilon}, \mathrm{H} : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_v \times n_s}$ and $\mathrm{K} : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_v \times n_\varepsilon}$ parameterized as

$$\begin{bmatrix} \mathrm{F}(\xi_m, y) & \mathrm{G}(\xi_m, y) \\ \mathrm{H}(\xi_m, y) & \mathrm{K}(\xi_m, y) \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{F}_0 & \mathrm{G}_0 \\ \mathrm{H}_0 & \mathrm{K}_0 \end{bmatrix} + \sum_{i=1}^{n_\lambda} \begin{bmatrix} \mathrm{F}_i & \mathrm{G}_i \\ \mathrm{H}_i & \mathrm{K}_i \end{bmatrix} \lambda_i(\xi_m, y) \tag{12}$$

where $\lambda : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_\lambda}$ is a free gain-scheduling function with arbitrary dimension $n_\lambda \in \mathbb{N}$ and $\mathrm{F}_0, \ldots, \mathrm{F}_{n_\lambda} \in \mathbb{R}^{n_s \times n_s}, \mathrm{G}_0, \ldots, \mathrm{G}_{n_\lambda} \in \mathbb{R}^{n_s \times n_\varepsilon}, \mathrm{H}_0, \ldots, \mathrm{H}_{n_\lambda} \in \mathbb{R}^{n_v \times n_s}$ and $\mathrm{K}_0, \ldots, \mathrm{K}_{n_\lambda} \in \mathbb{R}^{n_v \times n_\varepsilon}$ are free design terms to be synthesized. In comparison with [15], where a stability analysis method was proposed, this article is additionally accounting for the problem of systematically designing the aforementioned controller parameters, here considered to be *a priori* unknown.

B. Regulation Error System

We now introduce a change of state-space coordinates defined as $z \triangleq [z_x^T \; z_m^T]^T \triangleq [x^T - \pi^T(w) \;\; \xi_m^T - \sigma^T(w)]^T \in \mathbb{R}^{n_z} (n_z = n_x + n_m)$. Combining the system model (1) with the internal model functions (5), one can express the regulation error dynamics in the form

$$\begin{cases} \dot{z} = f_z(z,w,v) \\ \varepsilon = \delta_z(z,w) \end{cases} \tag{13}$$

where $f_z(z,w,v) \triangleq [f_x^T(z,w,v) \;\; f_m^T(z,w,v)]^T$ with $f_x(z,w,v) \triangleq f(z_x + \pi(w)\,,\, w\,,\, \theta_m(z_m + \sigma(w), g(z_x + \pi(w), w)) + v_u) - f(\pi(w), w, c(w))$ and $f_m(z,w,v) \triangleq \phi_m(z_m + \sigma(w), g(z_x + \pi(w), w)) - \phi_m(\sigma(w), d(w)) + v_m$. In turn, it also follows that $\delta_z(z,w) \triangleq \delta(g(z_x + \pi(w), w))$.

Matrix functions $\mathrm{F}(\xi_m, y), \ldots, \mathrm{K}(\xi_m, y)$, from the stabilizing stage, may also be expressed in terms of z and w by substituting $\lambda(\xi_m, y)$ in (12) with

$$\lambda(z,w) \triangleq \lambda(z_m + \sigma(w), g(z_x + \pi(w), w)). \tag{14}$$

The result of this substitution will be subsequently denoted as $F(z,w), \ldots, K(z,w)$.

Toward our main results, we consider that (13) admits a differential-algebraic representation [11] as follows.

Assumption 3: Nonlinear functions $f_z(z,w,v)$ and $\delta_z(z,w)$ can be represented as

$$\begin{aligned} f_z(z,w,v) &= A(z,w)\,z + \Phi(z,w)\,\varphi(z,w) + B\,v \\ \delta_z(z,w) &= C\,z + \Gamma\,\varphi(z,w) \end{aligned} \tag{15}$$

with a regular rational nonlinear function $\varphi : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi}$ satisfying

$$0 = \Psi(z,w)\,z + \Omega(z,w)\,\varphi(z,w) \tag{16}$$

such that:
1) sets $\mathcal{Z}^+$ and $\mathcal{W}^+$ satisfy $\{0\} \subset \mathcal{Z}^+ \subseteq \mathbb{R}^{n_z}$ and $\mathcal{W} \subset \mathcal{W}^+ \subseteq \mathbb{R}^{n_w}$;
2) matrices $A : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_z \times n_z}, \Phi : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_z \times n_\varphi}$, $\Psi : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi \times n_z}$ and $\Omega : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi \times n_\varphi}$ are affine with respect to (z,w);
3) matrices $B \in \mathbb{R}^{n_z \times n_v}, C \in \mathbb{R}^{n_\varepsilon \times n_z}$ and $\Gamma \in \mathbb{R}^{n_\varepsilon \times n_\varphi}$ are constant;
4) matrix $\Omega(z,w)$ is nonsingular $\forall\,(z,w) \in \mathcal{Z}^+ \times \mathcal{W}^+$;
5) for constant matrices $\mathrm{A}_0, \ldots, \mathrm{A}_{n_\lambda} \in \mathbb{R}^{n_z \times n_z}$, matrix $A(z,w)$ satisfies

$$A(z,w) = \mathrm{A}_0 + \sum_{i=1}^{n_\lambda} \mathrm{A}_i\,\lambda_i(z,w). \tag{17}$$

Remark 2: Representation (15), (16) can model the whole class of regular rational error functions from (13) inside $\mathcal{Z}^+$ and $\mathcal{W}^+$ [11]. Regular rational functions are those that can be expressed as a fraction of polynomial functions and that have no singularities in their domain.

Here, the validity region for the (z,w) variables is being specified by sets $\mathcal{Z}^+$ and $\mathcal{W}^+$, which are considered convex hull of vertices, i.e., $\mathcal{Z}^+ = \mathrm{Co}\{\mathcal{V}_z\}$ and $\mathcal{W}^+ = \mathrm{Co}\{\mathcal{V}_w\}$. Without loss of generality, $\mathcal{Z}^+$ is also required to be expressed in the polyhedral form

$$\mathcal{Z}^+ = \left\{ z \in \mathbb{R}^{n_z} : |\, p_k^T\, z\,| \le 1\,,\, k = 1, 2, \ldots, n_k \right\} \tag{18}$$

for given vectors $p_1, p_2, \ldots, p_{n_k} \in \mathbb{R}^{n_z}$.

Now let $\mathbf{z} \triangleq [z^T \;\; \xi_s^T]^T \in \mathbb{R}^{n_a}(n_a = n_z + n_s)$. Given the established differential-algebraic framework, one can write the combined dynamics of (11) and (13) as

$$\begin{cases} \dot{\mathbf{z}} = \boldsymbol{A}(z,w)\,\mathbf{z} + \boldsymbol{\Phi}(z,w)\,\varphi(z,w) \\ 0 = \boldsymbol{\Psi}(z,w)\,\mathbf{z} + \boldsymbol{\Omega}(z,w)\,\varphi(z,w) \end{cases} \tag{19}$$

where $\boldsymbol{\Omega}(z,w) = \Omega(z,w), \boldsymbol{\Psi}(z,w) = [\Psi(z,w) \;\; 0]$

$$\boldsymbol{A}(z,w) = \begin{bmatrix} A(z,w) + BK(z,w)C & BH(z,w) \\ G(z,w)C & F(z,w) \end{bmatrix}$$

$$\boldsymbol{\Phi}(z,w) = \begin{bmatrix} \Phi(z,w) + BK(z,w)\Gamma \\ G(z,w)\Gamma \end{bmatrix}. \tag{20}$$

Finally, the manifold attraction requirement (9) can now be expressed in the compact form

$$\lim_{t\to\infty} \|\mathbf{z}(t)\| = 0 \;\; \forall\, \big(\mathbf{z}(0), w(0)\big) \in \boldsymbol{\mathcal{D}} \subseteq \mathbb{R}^{n_a} \times \mathcal{W}. \tag{21}$$

C. Stability Conditions

To establish a systematic design procedure for $\mathrm{F}_0, \ldots, \mathrm{K}_{n_\lambda}$ in (12), one must develop conditions in which (21) is satisfied. In this sense, Theorem 1 shows our main result.

Theorem 1: Consider *a priori* given functions $\delta(y)$ and $\lambda(\xi_m, y)$ such that (10) is satisfied and (14) is a linear mapping with respect to (z,w). Suppose $n_s = n_z$ and there exist symmetric matrices $X, Y \in \mathbb{R}^{n_z \times n_z}$ and matrices $L \in \mathbb{R}^{n_\varphi \times n_\varphi}, \hat{\mathrm{F}}_0, \ldots, \hat{\mathrm{F}}_{n_\lambda} \in \mathbb{R}^{n_z \times n_z}, \hat{\mathrm{G}}_0, \ldots, \hat{\mathrm{G}}_{n_\lambda} \in \mathbb{R}^{n_z \times n_\varepsilon}, \hat{\mathrm{H}}_0, \ldots, \hat{\mathrm{H}}_{n_\lambda} \in \mathbb{R}^{n_v \times n_z}$ and $\hat{\mathrm{K}}_0, \ldots, \hat{\mathrm{K}}_{n_\lambda} \in \mathbb{R}^{n_v \times n_\varepsilon}$ such that $\forall\, k \in \{1, 2, \ldots, n_k\}, (z,w) \in \mathcal{V}_z \times \mathcal{V}_w$

$$\begin{bmatrix} X & I \\ \star & Y \end{bmatrix} \succ 0 \;\; \begin{bmatrix} 1 & p_k^T X & p_k^T \\ \star & X & I \\ \star & \star & Y \end{bmatrix} \succ 0 \tag{22}$$

$$\mathcal{H}\left\{ \begin{bmatrix} A(z,w)X + B\hat{H}(z,w) & A(z,w) + B\hat{K}(z,w)C & \Phi(z,w) + B\hat{K}(z,w)\Gamma \\ \hat{F}(z,w) & YA(z,w) + \hat{G}(z,w)C & Y\Phi(z,w) + \hat{G}(z,w)\Gamma \\ L\Psi(z,w)X & L\Psi(z,w) & L\Omega(z,w) \end{bmatrix} \right\} \prec 0\,. \tag{23}$$

Then the closed-loop system (1), (2) with controller (4), (11) achieves output regulation in

$$\mathcal{D} = \left\{ (x, \xi_m, \xi_s, w) \in \mathbb{R}^{n_a} \times \mathcal{W} \,:\, \mathbf{z}^T P\,\mathbf{z} \le 1 \right\} \tag{24}$$

$$P = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}^{-1} \tag{25}$$

with stabilizing gains $\mathrm{F}_i = N^{-1}(\hat{\mathrm{F}}_i + YB\hat{\mathrm{K}}_i CX - \hat{\mathrm{G}}_i CX - YB\hat{\mathrm{H}}_i - Y\mathrm{A}_i X)M^{-T}, \mathrm{G}_i = N^{-1}(\hat{\mathrm{G}}_i - YB\hat{\mathrm{K}}_i), \mathrm{H}_i = (\hat{\mathrm{H}}_i - \hat{\mathrm{K}}_i CX) M^{-T}, \mathrm{K}_i = \hat{\mathrm{K}}_i, i = 1, \ldots, n_\lambda$, where $M, N \in \mathbb{R}^{n_z \times n_z}$ are nonsingular solutions to $MN^T = I - XY$.

Proof: Consider the Lyapunov candidate function $V(\mathbf{z}) = \mathbf{z}^T P\,\mathbf{z}$. If matrix P is symmetric and positive-definite, then $V(\mathbf{z}) > 0 \;\forall\, \mathbf{z} \in \mathbb{R}^{n_a}, \mathbf{z} \neq 0$. The derivative of $V(\mathbf{z})$ along the trajectories of (19) is given by $\dot{V}(\mathbf{z}, w) = \mathcal{H}\{\mathbf{z}^T\,\boldsymbol{\Delta}_1(z,w)\,\zeta(\mathbf{z},w)\}$, where $\boldsymbol{\Delta}_1(z,w) \triangleq [P\boldsymbol{A}(z,w) \;\; P\boldsymbol{\Phi}(z,w)]$ and $\zeta(\mathbf{z},w) \triangleq [\mathbf{z}^T \;\; \varphi^T(z,w)]^T$. Similarly, equality constraint in (19) can be represented by $\boldsymbol{\Delta}_2(z,w)\,\zeta(\mathbf{z},w) = 0$, where $\boldsymbol{\Delta}_2(z,w) \triangleq [\boldsymbol{\Psi}(z,w) \;\; \boldsymbol{\Omega}(z,w)]$. If there exists a matrix $L \in \mathbb{R}^{n_\varphi \times n_\varphi}$ such that $\dot{V}(\mathbf{z},w) + \mathcal{H}\{\varphi^T(z,w)\,L\boldsymbol{\Delta}_2(z,w)\,\zeta(\mathbf{z},w)\} < 0 \forall\,(\mathbf{z},w) \in \boldsymbol{\mathcal{Z}}^+ \times \mathcal{W}^+$, $\mathbf{z} \neq 0$, then $\dot{V}(\mathbf{z},w) < 0 \forall\,(\mathbf{z},w) \in \boldsymbol{\mathcal{Z}}^+ \times \mathcal{W}^+, \mathbf{z} \neq 0$, for the set $\boldsymbol{\mathcal{Z}}^+ \triangleq \mathcal{Z}^+ \times \mathbb{R}^{n_s}$. By factorizing $\zeta(\mathbf{z},w)$, this last condition simplifies to

$$\mathcal{H}\left\{ \begin{bmatrix} P\boldsymbol{A}(z,w) & P\boldsymbol{\Phi}(z,w) \\ L\boldsymbol{\Psi}(z,w) & L\boldsymbol{\Omega}(z,w) \end{bmatrix} \right\} \prec 0 \tag{26}$$

$\forall\,(z,w)\in\mathcal{Z}^+\times\mathcal{W}^+$, or equivalently $\forall\,(z,w)\in\mathcal{V}_z\times\mathcal{V}_w$, since $\boldsymbol{A}(z,w),\ldots,\boldsymbol{\Omega}(z,w)$ are affine matrix functions. Conditions $P\succ 0$ and (26) so ensure that $V(\mathbf{z})$ is positive-definite and its derivative is negative-definite $\forall\,(\mathbf{z},w)\in\boldsymbol{\mathcal{Z}}^+\times\mathcal{W}^+$. The candidate domain of attraction estimate and positively invariant region of system (19) can then be expressed as $\boldsymbol{\mathcal{D}}=\boldsymbol{\mathcal{Z}}\times\mathcal{W}$, where $\boldsymbol{\mathcal{Z}}=\left\{\mathbf{z}\in\mathbb{R}^{n_a}:V(\mathbf{z})\le 1\right\}$. Since $w(0)\in\mathcal{W}\Rightarrow w(t)\in\mathcal{W}\subset\mathcal{W}^+\forall\,t\ge 0$, it is necessary to ensure $\boldsymbol{\mathcal{Z}}\subset\boldsymbol{\mathcal{Z}}^+$ to guarantee that $(\mathbf{z}(0),w(0))\in\boldsymbol{\mathcal{Z}}\times\mathcal{W}\Rightarrow(\mathbf{z}(t),w(t))\in\boldsymbol{\mathcal{Z}}^+\times\mathcal{W}^+\forall\,t>0$. Using the polyhedral definition of $\mathcal{Z}^+$ in (18), the inclusion $\boldsymbol{\mathcal{Z}}\subset\boldsymbol{\mathcal{Z}}^+$ is equivalent to [17]

$$\begin{bmatrix}1 & \boldsymbol{p}_k^T\\ \star & P\end{bmatrix}\succ 0\quad \boldsymbol{p}_k\triangleq\begin{bmatrix}p_k\\ 0\end{bmatrix}\tag{27}$$

$\forall\,k\in\{1,2,\ldots,n_k\}$. According to [18], the trajectory $\mathbf{z}(t)$ of the system (19) asymptotically converges to the origin for every initial condition $(\mathbf{z}(0),w(0))\in\boldsymbol{\mathcal{D}}$ if $P\succ 0$, (26) and (27) $\forall\,k\in\{1,2,\ldots,n_k\},\forall\,(z,w)\in\mathcal{V}_z\times\mathcal{V}_w$. From Lemma 1, it follows that the original closed-loop system (1), (2), (4), (11) achieves output regulation in the region defined by (24).

Consider now a full-order controller ($n_s=n_z$) and the following definitions [19]:

$$\begin{bmatrix}Y & N\\ N^T & \cdot\end{bmatrix}\triangleq P\quad\begin{bmatrix}X & M\\ M^T & \cdot\end{bmatrix}\triangleq P^{-1}$$

$$Z_1\triangleq\begin{bmatrix}X & I\\ M^T & 0\end{bmatrix},\ Z_2\triangleq\begin{bmatrix}I & Y\\ 0 & N^T\end{bmatrix}\tag{28}$$

noting that $PZ_1=Z_2$. Because $PP^{-1}=I$, then M,N are solutions to $MN^T=I-XY$. By post and premultiplying $P\succ 0$, (26), (27), respectively, by $Z_1,\mathrm{diag}\{Z_1,I\}$ and $\mathrm{diag}\{1,Z_1\}$, one obtains the conditions in (22) and (23) when using the change of variables: $\hat{F}(z,w)\triangleq YA(z,w)X+YBK(z,w)CX+NG(z,w)CX+YBH(z,w)M^T+NF(z,w)M^T,\hat{G}(z,w)\triangleq YBK(z,w)+NG(z,w),\hat{H}(z,w)\triangleq K(z,w)CX+H(z,w)M^T$, and $\hat{K}(z,w)\triangleq K(z,w)$. Inversion of these transformations yield the expressions in Theorem 1. ■

Remark 3: For sufficiently small sets $\mathcal{Z}^+\to\{z=0\},\mathcal{W}^+\to\{w=w_c\}$, where w_c is the center point of $\mathcal{W}^+$, the regulation error system from (15) can be simplified to $\dot{z}\approx A(0,w_c)\,z+B\,v$ and $\varepsilon\approx C\,z$. In this local context, conditions from Theorem 1 are solvable if and only if pairs $\{A(0,w_c),B\}$ and $\{A(0,w_c),C\}$ are stabilizible and detectable since our result becomes equivalent to [19]. This requisite is thus a necessary condition for the solvability of (22), (23) for larger sets $\mathcal{Z}^+\times\mathcal{W}^+\supset\{z=0,w=w_c\}$, where the nonlinear terms play a significant role in the system dynamics.

By assuming $\xi_s(0)=0$, without loss of generality, it follows that $\mathbf{z}^T(0)P\,\mathbf{z}(0)=z^T(0)Yz(0)$. The set of admissible initial states $\mathcal{D}$ from (24) is thus enlarged by minimizing $\mathrm{tr}(Y)$. The parameters of the stabilizing controller (5) should then be synthesized by solving the following optimization problem:

$$\min_{X,Y,L,\hat{\mathrm{F}}_0,\ldots,\hat{\mathrm{K}}_{n_\lambda}}\ \mathrm{tr}(Y)\ \text{ s.t. }\left\{(22)\,,\,(23)\right\}\tag{29}$$

$\forall\,k\in\{1,2,\ldots,n_k\},(z,w)\in\mathcal{V}_z\times\mathcal{V}_w$. Clearly, (29) is not necessarily convex due to the bilinearities involving the pair of decision variables L and X in (23). However, if either of these matrices is regarded as *a priori* fixed, then (29) becomes a standard semidefinite optimization. This idea can be employed in order to iteratively find a locally optimal solution, similar to the so-called D-K iteration method [20].

There exists a particular class of nonlinear functions $\varphi(z,w)$ where the synthesis problem is readily convex. For such, we consider the subsequent extra assumption.

Assumption 4: Assumption 3 holds with the following:

1) There exists a function $\varphi(y):\mathbb{R}^{n_y}\to\mathbb{R}^{n_\varphi}$ such that

$$\varphi(z,w)=\varphi(g(z_x+\pi(w),w)).\tag{30}$$

2) For constant matrices $\Phi_0,\ldots,\Phi_{n_\lambda}\in\mathbb{R}^{n_z\times n_\varphi}$, matrix $\Phi(z,w)$ satisfies

$$\Phi(z,w)=\Phi_0+\sum_{i=1}^{n_\lambda}\Phi_i\,\lambda_i(z,w).\tag{31}$$

What is being stated by Assumption 4 is that all rational nonlinearities $\varphi(z,w)$ can be exactly remapped as a function of y, i.e., $\varphi(y)$. In this particular case, we propose the implementation of the following modified nonlinear stabilizing controller:

$$\begin{cases}\dot{\xi}_s=\mathrm{F}(\xi_m,y)\,\xi_s+\mathrm{G}(\xi_m,y)\,\varepsilon+\Lambda(\xi_m,y)\,\varphi(y)\\ v=\mathrm{H}(\xi_m,y)\,\xi_s+\mathrm{K}(\xi_m,y)\,\varepsilon+\Theta(\xi_m,y)\,\varphi(y)\end{cases}\tag{32}$$

where terms $\Lambda:\mathbb{R}^{n_y}\to\mathbb{R}^{n_s\times n_\varphi}$ and $\Theta:\mathbb{R}^{n_y}\to\mathbb{R}^{n_v\times n_\varphi}$ are additional free design matrix functions constructed similarly as (12). Corollary 1 in the sequel addresses the output regulation problem based on controller (32).

Corollary 1: Consider *a priori* given functions $\delta(y),\varphi(y)$, and $\lambda(\xi_m,y)$ such that (10) and (30) are satisfied and (14) is a linear mapping with respect to (z,w). Suppose $n_s=n_z$ and there exist symmetric matrices $X,Y\in\mathbb{R}^{n_z\times n_z}$ and matrices $\hat{L}\in\mathbb{R}^{n_\varphi\times n_\varphi},\hat{\mathrm{F}}_0,\ldots,\hat{\mathrm{F}}_{n_\lambda}\in\mathbb{R}^{n_z\times n_z},\hat{\mathrm{G}}_0,\ldots,\hat{\mathrm{G}}_{n_\lambda}\in\mathbb{R}^{n_z\times n_\varepsilon},\hat{\mathrm{H}}_0,\ldots,\hat{\mathrm{H}}_{n_\lambda}\in\mathbb{R}^{n_v\times n_z},\hat{\mathrm{K}}_0,\ldots,\hat{\mathrm{K}}_{n_\lambda}\in\mathbb{R}^{n_v\times n_\varepsilon},\hat{\Lambda}_0,\ldots,\hat{\Lambda}_{n_\lambda}\in\mathbb{R}^{n_z\times n_\varphi}$ and $\hat{\Theta}_0,\ldots,\hat{\Theta}_{n_\lambda}\in\mathbb{R}^{n_v\times n_\varphi}$ such that (22) $\forall\,k\in\{1,2,\ldots,n_k\}$ and $\forall\,(z,w)\in\mathcal{V}_z\times\mathcal{V}_w$

$$\mathcal{H}\left\{\begin{bmatrix}\begin{matrix}A(z,w)X+\\ B\hat{H}(z,w)\end{matrix} & \begin{matrix}A(z,w)+\\ B\hat{K}(z,w)C\end{matrix} & \begin{matrix}\Phi(z,w)\hat{L}+\\ B\hat{\Theta}(z,w)\end{matrix}\\ \hat{F}(z,w) & \begin{matrix}YA(z,w)+\\ \hat{G}(z,w)C\end{matrix} & \hat{\Lambda}(z,w)\\ \Psi(z,w)X & \Psi(z,w) & \Omega(z,w)\hat{L}\end{bmatrix}\right\}\prec 0\,.\tag{33}$$

Then the closed-loop system (1), (2) with controller (4), (32) achieves output regulation in region (24) with stabilizing parameters $\mathrm{F}_i,\ldots,\mathrm{K}_i$ obtained as in Theorem 1 and with $\Lambda_i=N^{-1}\big((\hat{\Lambda}_i-YB\hat{\Theta}_i)\hat{L}^{-1}+(YB\hat{\mathrm{K}}_i-\hat{\mathrm{G}}_i)\Gamma-Y\Phi_i\big)$ and $\Theta_i=\hat{\Theta}_i\hat{L}^{-1}-\hat{\mathrm{K}}_i\Gamma,i=1,\ldots,n_\lambda$, where $M,N\in\mathbb{R}^{n_z\times n_z}$ are nonsingular solutions to $MN^T=I-XY$.

Proof: Follow the same steps presented in the proof of Theorem 1, however, post and premultiply (26) by $\mathrm{diag}\{Z_1,L^{-T}\}$ and its transpose, which yields (33) instead of (23) considering the additional change of variables $\hat{L}=L^{-T},\hat{\Lambda}(z,w)=(Y\Phi(z,w)+YB\Theta(z,w)+YBK(z,w)\Gamma+N\Lambda(z,w)+NG(z,w)\Gamma)L^{-T}$ and $\hat{\Theta}(z,w)=(\Theta(z,w)+K(z,w)\Gamma)L^{-T}$. ■

The synthesis problem may now be addressed by the following numerical optimization:

$$\min_{X,Y,\hat{L},\hat{\mathrm{F}}_0,\ldots,\hat{\Theta}_{n_\lambda}}\ \mathrm{tr}(Y)\ \text{ s.t. }\left\{(22)\,,\,(33)\right\}\tag{34}$$

$\forall\,k\in\{1,2,\ldots,n_k\},(z,w)\in\mathcal{V}_z\times\mathcal{V}_w$, which is always convex because (33) is now linear with respect to all decision variables.

IV. Numerical Example

Consider a rational nonlinear plant and a chaotic Lorenz exosystem [21]

$$\begin{cases} \dot{x}_1 = x_1^2\,(1+x_1^2)^{-1} + x_2 + a_1\,w_2 \\ \dot{x}_2 = x_1\,(1+x_1^2)^{-1} + a_1\,w_1(1+w_3) + a_2\,u \\ y_1 = x_1 + a_3\,x_2 x_1\,(1+x_1^2)^{-1} \\ y_2 = w_1 \end{cases} \tag{35}$$

$$\begin{cases} \dot{w}_1 = b_1\,(w_2 - w_1) \\ \dot{w}_2 = b_2\,w_1 - w_2 - b_4\,w_1 w_3 \\ \dot{w}_3 = b_4\,w_1 w_2 - b_3\,w_3 \end{cases} \tag{36}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, b_4 \in \mathbb{R}$ are constant parameters, and suppose the target output error is $e = x_1$. The plant zero-error steady state is thus described by $\pi(w) = [0 \;\; -a_1\,w_2]^T$ and $c(w) = a_1 a_2^{-1}(w_2 - \tilde{b}_2\,w_1 + \tilde{b}_4\,w_1 w_3)$, where $\tilde{b}_2 \triangleq b_2 + 1$ and $\tilde{b}_4 \triangleq b_4 - 1$. A feasible internal model for this setup is

$$\begin{cases} u = \xi_{m2} - \tilde{b}_2\,\xi_{m1} + \tilde{b}_4\,y_2\,\xi_{m3} + v_1 \\ \dot{\xi}_{m_1} = b_1\,(\xi_{m2} - \xi_{m1}) + v_2 \\ \dot{\xi}_{m_2} = b_2\,\xi_{m1} - \xi_{m2} - b_4\,y_2\,\xi_{m3} + v_3 \\ \dot{\xi}_{m_3} = b_4\,y_2\,\xi_{m2} - b_3\,\xi_{m3} + v_4 \end{cases} \tag{37}$$

where $\sigma(w) = a_1 a_2^{-1}\,w$. In order to provide the remaining stabilizing inputs $v \in \mathbb{R}^4$, we exemplify the proposed framework subsequently.

As a preliminary setup, it is considered $\delta(y) = y_1$ and $\lambda(y) = y_2$, as the stabilizing reference and gain-scheduling parameter, respectively. By then applying the change of state-space coordinates indicated in Section III-B, the system equations in the form $\dot{z} = f_z(z, w, v)$ and $\varepsilon = \delta_z(z, w)$ is obtained as follows:

$$\begin{cases} \dot{z}_1 = z_1^2\,(1+z_1^2)^{-1} + z_2 \\ \dot{z}_2 = z_1\,(1+z_1^2)^{-1} + a_2\,(z_4 - \tilde{b}_2\,z_3 + \tilde{b}_4\,w_1 z_5 + v_1) \\ \dot{z}_3 = b_1\,(z_4 - z_3) + v_2 \\ \dot{z}_4 = b_2\,z_3 - z_4 - b_4\,w_1 z_5 + v_3 \\ \dot{z}_5 = b_4\,w_1 z_4 - b_3\,z_5 + v_4 \\ \varepsilon = z_1 + a_3\,(z_1 z_2 - a_1\,z_1 w_2)\,(1+z_1^2)^{-1}. \end{cases} \tag{38}$$

In this example, it suffices to use the vector of rational nonlinearities

$$\varphi(z, w) = (1+z_1^2)^{-1}\begin{bmatrix} z_1^2 & z_1 & z_1 z_2 & z_1 w_2 \end{bmatrix}^T \tag{39}$$

in order to express system (38) as in Assumption 3. We additionally define $\mathcal{W}^+ = \mathrm{Co}\{-\overline{r}\,,\,\overline{r}\} \times \mathrm{Co}\{-\overline{r}\,,\,\overline{r}\} \times \mathbb{R} \supset \mathcal{W}$ and $\mathcal{Z}^+ = \mathrm{Co}\{-\overline{z}_1\,,\,\overline{z}_1\} \times \mathrm{Co}\{-\overline{z}_2\,,\,\overline{z}_2\} \times \mathbb{R}^3$. Parameters $\overline{z}_i > 0$ denote the maximum admissible value of $|z_i(t)| \forall\, t \geq 0$, while $\overline{r} \triangleq (b_1 + b_2) b_3 (2 b_4 \sqrt{b_3 - 1})^{-1}$ is the radius of a positively invariant ball $\mathcal{W}$ where the chaotic trajectory of the Lorenz exosystem is contained [21].

Considering $a_1 = 1, a_2 = 10^3, a_3 = 10^{-5}, b_1 = 10, b_2 = 28, b_3 = 8/3, b_4 = 1, \overline{z}_1 = 10^2$, and $\overline{z}_2 = 10^4$, optimization problem (29) was iteratively evaluated in order to synthesize the parameters for stabilizing controller (11), resulting an approximately locally optimal solution with objective value $\mathrm{tr}(Y) = 1.65 \cdot 10^{-3}$. Fig. 1 shows the output error signal $e(t)$ and the control input $u(t)$ obtained by a numerical simulation of the closed loop with this designed controller.

Now suppose that $a_3 = 0$, which implies the first measurement of plant (35) is directly $y_1 = x_1$. In this particular case, it suffices to consider a simpler vector of rational nonlinearities in order to represent the system (38) as in Assumption 3

$$\varphi(z) = (1+z_1^2)^{-1}\begin{bmatrix} z_1^2 & z_1 \end{bmatrix}^T. \tag{40}$$

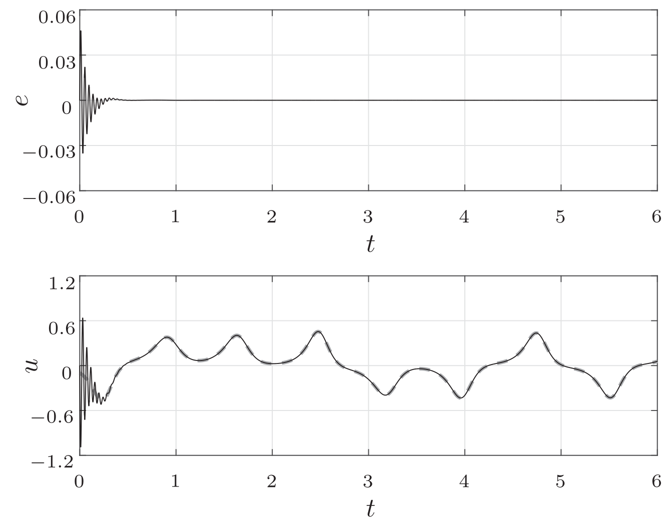

Fig. 1. Output error e and control input u using stabilizing controller (11) synthesized by (29). The zero-error steady-state $c(w)$ is shown in dashed-line.

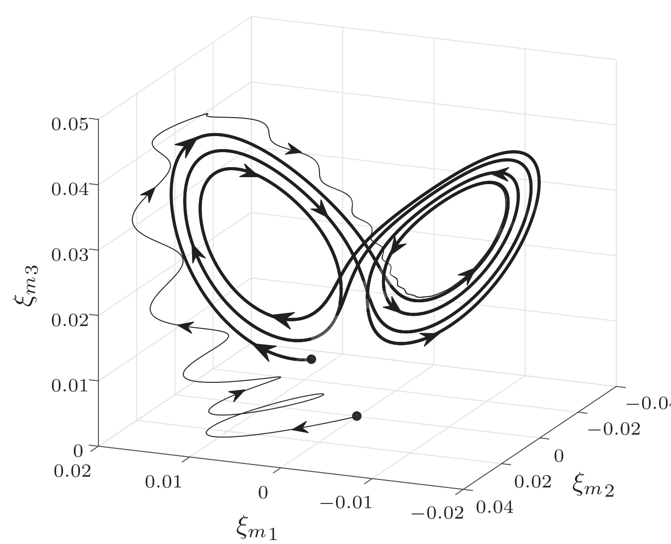

Fig. 2. Internal model trajectory ξ_m compared to the zero-error steady-state $\sigma(w)$ (thick line) using stabilizing controller (32) synthesized by (34).

Since $z_1 = y_1$, it verifies that $\varphi(z)$ in (40) can be remapped as $\varphi(y) = (1+y_1^2)^{-1}[y_1^2 \;\; y_1]^T$, therefore the complementary Assumption 4 holds. According to Corollary 1, it is now possible to implement the nonlinear stabilizing stage (32) and the synthesis can be addressed by a single convex optimization problem, as indicated by (34). Using the same numerical setup from the general case previously mentioned, semidefinite programming (34) yielded an optimal objective value of $\mathrm{tr}(Y) = 1.12 \cdot 10^{-3}$. A simulated trajectory of the closed-loop system with the synthesized controller (32) is shown in Fig. 2. This graphical representation compares the internal model trajectory ξ_m with respect to the target zero-error steady-state trajectory $\sigma(w) = a_1 a_2^{-1}\,w$.

V. Conclusion

This article presented a methodology for designing dynamic output feedback stabilizing controllers for the output regulation problem. The devised method is based on the differential-algebraic representation and able to systematically address the large class of nonlinear systems with rational regulation error dynamics. The resulting control design is cast as an optimization problem subject to bilinear matrix constraints. A special case was also presented where the constraints become linear and the optimization problem convex.

This article also sets the stage to a wide range of future works, such as the extension to saturating control inputs, multiagent systems, and cooperative control [22].

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